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Best Simultaneous L_1 Approximations

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Three possible definitions are proposed for best simultaneous L_1 approximation to n continuous real-valued functions, and the relation between best simultaneous approximations and best L_1 approximations to the arithmetic mean of the n functions is discussed.

1

Several authors have considered best simultaneous approximations to two functions f_1 and f_2 belonging to $C[a, b]$ by elements of a subset S of $C[a, b]$. Diaz and McLaughlin [2, 3] and Ling [5] have considered best approximations in the supremum norm and Phillips and Sahney [7] have given results for the L_1 and L_2 norms. The problem of best simultaneous approximation to an arbitrary number of functions has been discussed by Holland and Sahney [4], who have generalized the results in [7] for the L_2 norm, and by Cheney, McCabe, and Phillips [6] who have generalized Ling's [5] work using the supremum norm.

2

In each of the papers cited above, a definition of best simultaneous approximation is given and a result of the following kind is established: the best simultaneous approximation to n ($\geqslant 2$) given functions coincides with the best approximation (in the relevant norm, but with an important modification in the case of [3]) to the arithmetic mean of the n functions.

We now examine three possible definitions of best simultaneous L_1 approximation to n functions and explore whether, for any of these defi-

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nitions, the best simultaneous approximation coincides with the best L_1 approximation to the mean of the n functions.

DEFINITION 1. Given elements $f_1, f_2, ..., f_n$ of $C[a, b]$ and S a subset of $C[a, b]$, we say that $s^* \in S$ is a best simultaneous L_1 approximation to $f_1, f_2, ..., f_n$ if

$$\max_j \| f_j - s^* \| \leqslant \max_j \| f_j - s \|$$

for all $s \in S$, where $\| \cdot \|$ denotes the L_1 norm on $C[a, b]$.

DEFINITION 2. Given elements $f_1, f_2, ..., f_n$ of $C[a, b]$ and S a subset of $C[a, b]$, we say that s^* is a best simultaneous L_1 approximation to $f_1, f_2, ..., f_n$ if

$$\int_a^b \max_j | f_j(x) - s^*(x) | \, dx \leqslant \int_a^b \max_j | f_j(x) - s(x) | \, dx$$

for all $s \in S$.

DEFINITION 3. Given elements $f_1, f_2, ..., f_n$ of $C[a, b]$ and S a subset of $C[a, b]$, we say that s^* is a best simultaneous L_1 approximation to $f_1, f_2, ..., f_n$ if

$$\sum_{i=1}^{n} \| f_i - s^* \| \leqslant \sum_{i=1}^{n} \| f_i - s \|$$

for all $s \in S$.

Remark. Phillips and Sahney [7] showed that the best simultaneous approximation to two functions in the sense of Definition 2 does coincide with the best L_1 approximation to the arithmetic mean of the two functions.

3

In this section we consider best simultaneous L_1 approximations in the sense of Definition 1 above. First we note that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} f_i - s \right\| = \left\| \sum_{i=1}^{n} \frac{1}{n} (f_i - s) \right\| \leqslant \max_j \| f_j - s \|.$$

On taking the infimum over S, we find that the "error" of the best L_1 approximation to the mean is bounded above by the "error" of best simultaneous approximation in the sense of Definition 1. The following counterexample shows that, in general, the best simultaneous approximation in the sense of Definition 1 does *not* coincide with the best L_1 approximation to the mean.

COUNTEREXAMPLE 1. Choose $f_1(x) \equiv 0$ and $f_2(x) \equiv x$ on $[0, 1]$ and let S be the set of real numbers. A simple calculation shows that the best simultaneous approximation to f_1 and f_2 from S in the sense of Definition 1 is the number $1 - 1/(2)^{1/2}$, whereas the best L_1 approximation to $\frac{1}{2}(f_1 + f_2)$ is the number $\frac{1}{4}$.

4

We now consider best simultaneous approximation in the sense of Definition 2. First we quote a theorem of Phillips and Sahney [7].

THEOREM 1. *Let f_1 and f_2 be elements of $C[a, b]$ and S be a subset of $C[a, b]$. Then $s^* \in S$ is a best simultaneous approximation to f_1 and f_2 in the sense of Definition 2 if and only if it is a best L_1 approximation to $\frac{1}{2}(f_1 + f_2)$.*

We now use this theorem to show that it does not extend directly to more than two functions.

THEOREM 2. *If s^* is a best L_1 approximation to $(1/n) \sum_{i=1}^{n} f_i$ from S and $n > 2$, then in general s^* is not a best simultaneous approximation to f_1, f_2,..., f_n in the sense of Definition 2.*

Proof. Let f_1 and f_2 be arbitrary elements of $C[a, b]$ and let $f_j = f_2$ for $j \geqslant 2$. Then

$$\inf_{s \in S} \int_a^b \max_j |f_j(x) - s(x)| \, dx$$
$$= \inf_{s \in S} \int_a^b \max[|f_1(x) - s(x)|, |f_2(x) - s(x)|] \, dx.$$

By Theorem 1 above, the latter infimum is attained for $s = s^*$, the best L_1 approximation to $\frac{1}{2}(f_1 + f_2)$. In general, this s^* will not be the best L_1 approximation to

$$\frac{1}{n}(f_1 + f_2 + \cdots + f_n) = \frac{1}{n} f_1 + \frac{n-1}{n} f_2$$

for $n > 2$, and this completes the proof.

To obtain a result for the approximation of n functions, $n > 2$, in the sense of Definition 2, we define

$$g_1(x) = \max_k \{f_k(x), k = 1, 2,..., n\},$$
$$g_2(x) = \min_k \{f_k(x), k = 1, 2,..., n\}$$

and state:

THEOREM 3. *Let* $f_1, f_2, \ldots, f_n$ *be elements of* $C[a, b]$ *and* S *be a subset of* $C[a, b]$. *Then* $s^* \in S$ *is a best simultaneous approximation to* $f_1, f_2, \ldots, f_n$ *in the sense of Definition* 2 *if and only if it is a best approximation to* g_1 *and* g_2 *in the sense of Definition* 2.

Proof. For an arbitrary fixed x it is clear that

$$\max_k |f_k(x) - s(x)| = \max[|g_1(x) - s(x)|, |g_2(x) - s(x)|]$$

and the theorem follows on integrating both sides and taking the infimum over S.

The following theorem then follows from Theorems 3 and 1.

THEOREM 4. *Let* $f_1, f_2, \ldots, f_n$ *be elements of* $C[a, b]$ *and* S *be a subset of* $C[a, b]$. *Then* $s^* \in S$ *is a best simultaneous approximation to* $f_1, f_2, \ldots, f_n$ *in the sense of Definition* 2 *if and only if it is a best* L_1 *approximation to the arithmetic mean of* $\max_k\{f_k(x)\}$ *and* $\min_k\{f_k(x)\}$.

Remark. Note that, for $n = 2$,

$$\tfrac{1}{2} \max_k \{f_k(x)\} + \tfrac{1}{2} \min_k \{f_k(x)\} = \tfrac{1}{2}[f_1(x) + f_2(x)]$$

and we observe that Theorem 4 is a generalization of Theorem 1. We also note the similarity to the work of Diaz and McLaughlin [2] on simultaneous approximation in the supremum norm.

5

In this section we discuss best simultaneous L_1 approximation in the sense of Definition 3. We state:

THEOREM 5. *If* $\operatorname{sign}(s(x) - f_j(x))$ *is always positive (or always negative) for all* $x \in [a, b]$, *for all* $j = 1, 2, \ldots, n$ *and for all* $s \in S$, *then the best simultaneous approximation to* $f_1, f_2, \ldots, f_n$ *in the sense of Definition* 3 *coincides with the best* L_1 *approximation to the arithmetic mean of* $f_1, f_2, \ldots, f_n$.

Proof. From the hypotheses in the statement of the theorem,

$$\int_a^b \sum_{i=1}^n |f_i(x) - s(x)| \, dx = \int_a^b \left| \sum_{i=1}^n (f_i(x) - s(x)) \right| dx$$

$$= n \int_a^b \left| \frac{1}{n} \sum_{i=1}^n f_i(x) - s(x) \right| dx$$

and the proof is completed by taking the infimum over S.

We now give a counterexample to show that the best simultaneous approximation to f_1 and f_2 in the sense of Definition 3 does not, *in general*, coincide with the best L_1 approximation to the mean.

COUNTEREXAMPLE 2. Choose f_1, f_2, and S as in Counterexample 1. A simple computation shows that the best simultaneous approximation to f_1 and f_2, in the sense of Definition 3, is the constant function $s = 0$, whereas the best L_1 approximation to $\frac{1}{2}(f_1 + f_2)$ is $s = \frac{1}{4}$.

Remark. The conditions of Theorem 5 arise naturally in the study of one-sided approximations (see, for example, [1]). Further, Counterexample 2 shows the necessity of such conditions.

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